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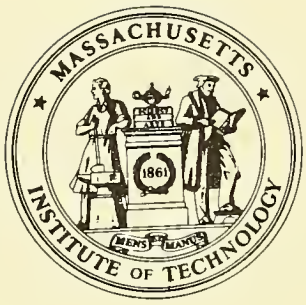

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CAPITAL GAINS, INCOME, AND SAVING

by

Karl Shell, Miguel Sidrauski, and Joseph E. Stiglitz

Number 12 -- December 1967

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The views expressed in this paper are the authors' sole responsibility, and do not reflect those of the Department of Economics, nor the Massachusetts Institute of Technology.

1. Introduction

Since the General Theory, the consumption function, relating the level of consumption to "income," has been a standard part of macroeconomic theory. In the usual analysis, the measure of "income" that is employed in the determination of the level of consumption is Disposable Income. Why is this chosen as the relevant measure of income? Presumably because the consumption function is derived from an aggregation over individual demands, and individuals base their decisions upon a microeconomic measure of income that is similar to the macroeconomic Disposable Income. But if this is in fact how the consumption function is derived, i.e., if it results from an aggregation over individual demands,[1] then income of individuals is not correctly

---

[1] On the other hand, the consumption function could be thought to be the result of government action. As in Solow's model [12], fiscal and monetary policy might be such that the level of savings (and investment) is a given fraction of Disposable Income. But this is a peculiar policy for the government to pursue, and this approach obviates at least the short-run macroeconomic analysis, which attempts to separate government activity from individual activity.

represented by Disposable Income, for capital gains are excluded. Individuals can consume or save out of capital gains just as they can consume or save out of wage or interest income. Indeed, if capital gains could be perfectly foreseen (and there were equal tax treatment), then individuals would be indifferent between interest income and capital gains. That tax advantages can make a dollar of expected capital gains preferable to a dollar of dividend or interest income for some individuals is evidence that expected capital gains do enter individuals' calculations. One individual buys IBM shares and receives a return in the form of capital gains; another individual buys AT&T shares and receives a return in the form of dividends. Clearly the former is as much a part of "income" as the latter.

We suggest that when expectations about price changes are fulfilled, the measure of income that is relevant for the determination of consumption is Disposable Income plus capital gains.[1] We call this definition of income Individual Purchasing Power (≡ Disposable Income + Capital Gains). Thus, Individual Purchasing Power is equal to the value of consumption that is consistent with zero change in the value of individuals' wealth. This definition of income has a long tradition; it is that proposed in 1921 by Robert Murray Haig [4]: consumption plus the change in the value of

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[1] When expectations are not fulfilled, that part of capital gains which is foreseen ought to be included; that part which is not foreseen has to be treated differently.

(individuals') wealth.[1] The Department of Commerce definition of Disposable Income is equivalent to consumption plus the value of the change in wealth. The difference between the two is capital gains.[2]

We do not want to embark here into a discussion of whether in fact savings as a function of income, properly defined, is a reasonable description of how the individual (or the economy) behaves. In the context of a dynamic economy, it is clear that if perfect borrowing and lending markets exist that wealth is a much better index of individual welfare than income.[3] Nor do we want to discuss whether individuals behave according to "rules" (like consumption a fixed fraction of "income") or whether they maximize their intertemporal utilities. We do want to argue, however, that if consumption is considered to be the result of aggregating over individuals, then the relevant definition of income is what we have called Individual Purchasing Power.

---

[1] In U. S. personal income taxation there are two important violations of the doctrine that taxation depends upon the Haig (and our!) measure of income: (1) Capital gains are taxed only when "realized," and (2) The rate of taxation on realized capital gains is lower than the rate of taxation on wages and dividends. The first violation is rationalized in terms of ease of administration, etc., while the second violation is rationalized as encouragement to risk-bearing. The recent Report of the Royal Commission on Taxation essentially proposes that Canada's personal income taxation be based upon Haig's definition.

[2] This is correct only if the value of human wealth is excluded. Since markets for "capitalizing" human wealth are notoriously imperfect, this assumption may not be unacceptable.

[3] Cf. Samuelson [9].

The purpose of this paper is to consider the implications for the traditional models of economic growth of this reformulation of the consumption function. First, we consider the two-sector growth model proposed by Meade and more fully articulated by Uzawa. Although long-run balanced growth is unchanged, the dynamics are considerably altered by our reformulation. We show how the dynamics of the Solow-Swan, one-sector economy are altered, although again long-run balanced growth is unaffected. When, however, a second store of value, for example, government debt, is introduced, not only are the dynamics affected but also the long-run equilibrium may be altered. Moreover, in none of the models that we have treated is dynamic behavior invariant to "the choice of numeraire"; the model in which the consumption value of savings is a constant fraction of Purchasing Power measured in consumption units is quite different from the model in which the capital goods value of savings is a constant fraction of Purchasing Power measured in capital goods units--and so forth.

2. The Two-Sector Model

The market value W of the existing stock of capital in terms of consumption goods is defined by

$$W = pK, \tag{2.1}$$

where K is the capital stock, and p is the market price of capital in terms of consumption. In the introduction, we argued that in an economy in which individuals are indifferent between returns in the form of factor rewards and in the form of capital gains, the appropriate definition of income from an individual point of view is the value of factor rewards plus expected capital gains due to expected changes in the relative prices less the value of the depreciation of the stock of physical capital. We have called this measure of income Individual Purchasing Power and now denote it by $\hat{Y}$ so that

$$\hat{Y} = Y_c + pY_I - p\mu K + \dot{p}^e K = Y - p\mu K + \dot{p}^e K, \tag{2.2}$$

where Y_c and Y_I are respectively the output of consumption and investment, Y is the consumption value of output (equal to the value of factor rewards), $\mu \geq 0$ is the constant relative rate of depreciation of capital, and $\dot{p}^e$ is the expected rate of change in the consumption price of capital. If expectations about price changes are always realized, $\dot{p}^e = \dot{p}$, and if individuals attempt to save a constant fraction s of Purchasing Power, then from (2.2) demand for savings is

$$s[Y + \dot{p}K - p\mu K].$$

Realized savings are equal to the change in the value of wealth,

$$\dot{W} = p\dot{K} + \dot{p}K. \tag{2.3}$$

In market equilibrium, then

$$(2.4) \qquad \dot{p}K + K\dot{p} = s[Y + \dot{p}K - p\dot{K}],$$

with $0 < s < 1$. Our savings hypothesis is consistent with that of Uzawa[1] only when individuals expect that prices will not change and that capital will not depreciate.

Momentary Equilibrium. Denote per capita quantities by lower-case letters. The heavy curve in Figure 1 is the production possibility frontier (PPF) corresponding to the value of the capital-labor ratio k. If competitive producers maximize profits, then given k, the price ratio p determines per capita output of investment and consumption, $y_I(k,p)$ and $y_c(k,p)$. If capital intensities are unequal, along the PPF, y_I is a strictly concave function of y_c. Therefore, given k, if efficient capital intensities in the two sectors differ, the per capita composition of output is uniquely determined by p. If capital intensities are equal, the PPF is a straight-line segment and the per capita composition of output is uniquely determined if and only if p is unequal to minus the slope of the (straight line)

---

[1] In [14], Uzawa assumes that gross savings are a constant fraction of the value of gross national product Y.

The discrete time analogue to equation (2.4) is $p_t(K_{t+\Delta t} - K_t) + (p_{t+\Delta t} - p_t)K_t = s[\Delta t Y(p_t, K_t, L_t) - \Delta t p_{t+\Delta t} K_t + (p_{t+\Delta t} - p_t)K_t]$, where $Y(p_{t+\Delta t}, K_t, L_t)$ is the consumption value of the average rate of output produced in the period $[t, t+\Delta t]$. Dividing by Δt and taking the limit as $t \to 0$ yields (2.4).

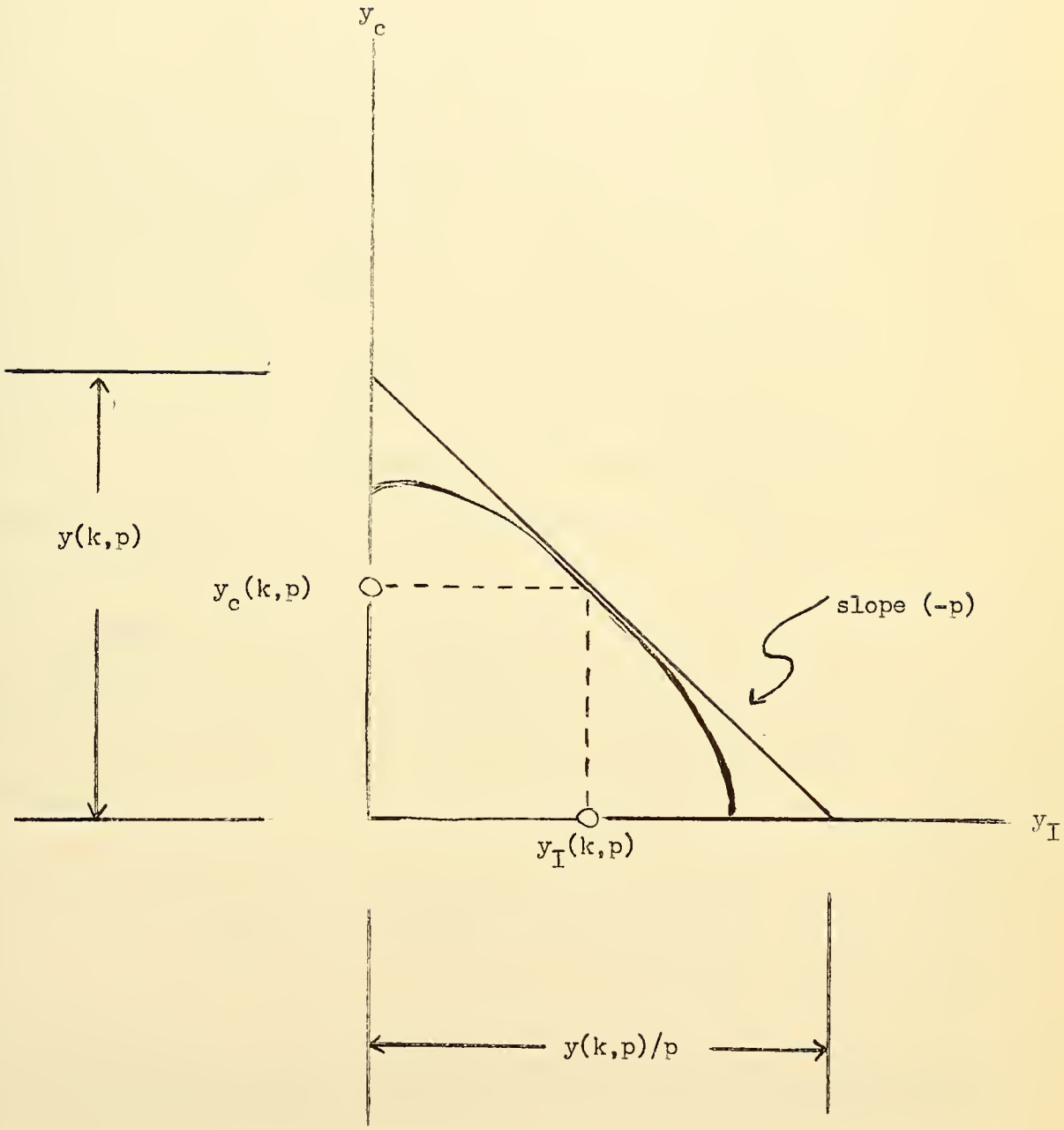

Figure 1

PPF.[1]

If capital intensities differ, the labor force L, the capital stock K, and the price ratio p uniquely determine Y_I and Y_c and hence $Y = Y_c + pY_I$ and $\dot{K} = Y_I - \mu K$. Therefore (2.4) can be solved for the unique $\dot{p}$ which will clear the savings-investment market, given the inherited endowments of capital and labor, and the inherited price ratio. Notice then a major difference between our formulation of the two-sector model and that of the previous two-sector models. In the previous models, determination of momentary equilibrium (given factor endowments) involves finding a price ratio that will clear the market. In our model, however, the economy inherits prices as well as endowments. Determination of momentary equilibrium involves finding the rate of change in the price ratio that will equate supply and demand.

Dynamics. As usual, we assume that labor L is growing at the exogenously given relative rate n. For ease of exposition, we assume that the rate of depreciation $\mu = 0$.[2] Then differential equation (2.4) can be rewritten as

$$\dot{pK}/L = sy - (1-s)\dot{pk}. \tag{2.5}$$

---

[1] Even though momentary equilibrium may not be uniquely determined, the system is always causal. Causality will be discussed in more detail in the next section on the one-sector model.

[2] This latter assumption will be relaxed in section 4. Indeed, combining the assumption $\mu > 0$ with $n = 0$ will have important welfare implications for the economy that follows the savings rule (2.4).

When the supply and demand for investment goods are equal $\dot{K} = Y_I$, and (2.5) reduces to

$$\dot{p} = \frac{sy(k,p) - py_I(k,p)}{(1-s)k} = \frac{p\left[(w+rk)s/p - y_I(k,p)\right]}{(1-s)k} \tag{2.6}$$

where w is the wage rate, r is the rentals rate on capital, and $y = w + rk$. Since $k \equiv K/L$ and $\dot{K}/L = y_I$, capital accumulation is given by

$$\dot{k} = y_I(k,p) - nk. \tag{2.7}$$

Differential equations (2.6) and (2.7) completely describe the laws of motion in our two-sector economy. k and p uniquely determine y and y_I and hence uniquely determine $\dot{p}$ and $\dot{k}$.

In order to simplify the analysis, in what follows we assume that the consumption goods industry is always more capital intensive than the investment goods industry. A balanced growth path is defined by $\dot{p} = 0 = \dot{k}$. Since there are no capital gains in balanced growth, our balanced growth state is identical to that of Uzawa [14]. If production functions satisfy the Inada conditions and the capital intensity hypothesis, then there exists a nontrivial balanced growth state.[1] That is, borrowing from Uzawa [14], we know that there exists a positive (k^*,p^*) such that

[1] See [14], pp. 111-112. It is important to notice, however, that out of balanced growth, the dynamics of our model differs from that in [14]. The proof of the existence of a balanced growth path uses the Inada assumptions about the production functions.

$$sy(k^*,p^*) = p^* y_I(k^*,p^*)$$

and

$$y_I(k^*,p^*) = nk^*.$$

We shall now prove that in our model, if the capital intensity hypothesis is satisfied, the nontrivial balanced growth equilibrium (k^*,p^*) is locally stable. Taking the linear Taylor approximation[1] of the system of differential equations (2.6) and (2.7) about (k^*,p^*) yields

$$(2.8) \qquad \begin{bmatrix} \dot{k} \\ \dot{p} \end{bmatrix} \doteq \begin{bmatrix} \frac{\partial y_I}{\partial k} - n & \frac{\partial y_I}{\partial p} \\ \frac{p}{(1-s)k}\left(\frac{sr}{p} - \frac{\partial y_I}{\partial k}\right) & \frac{p}{(1-s)k}\left(\frac{\partial (sy/p)}{\partial p} - \frac{\partial y_I}{\partial p}\right) \end{bmatrix} \begin{bmatrix} (k - k^*) \\ (p - p^*) \end{bmatrix}$$

where the expressions in the 2x2 matrix are evaluated at (k^*,p^*). If x is a characteristic root corresponding to the system (2.8), then

---

[1] It is a well-known proposition in two-sector theory (see e.g. [14]) that, when the capital intensity hypothesis is satisfied and when both goods are produced, the price ratio p uniquely determines the wage-rentals ratio which in turn uniquely determines the respective capital intensities in the two sectors; thus uniquely determining the wage rate w and the rentals rate r independent of k. Thus $\partial r/\partial k = 0 = \partial w/\partial k$.

$$x^2 - \left[\frac{p}{(1-s)k} \frac{\partial(sy/p)}{\partial p} - \frac{\partial y_I}{\partial p} + \frac{\partial y_I}{\partial k} - n \right] x$$

$$+ \frac{p}{(1-s)k} \left[\frac{\partial y_I}{\partial p} (n - sr/p) + \left(\frac{\partial y_I}{\partial k} - n \right) \frac{\partial(sy/p)}{\partial p} \right] = 0.$$

It follows immediately from Figure 1 that $\left(\frac{\partial y_I}{\partial p}\right) > 0$ and $\frac{\partial(sy/p)}{\partial p} < 0$. At (k^*,p^*), $n = y_I/k = s(w/pk + r/p) > sr/p$. And from Rybczynski's theorem,[1] we know that, under our capital intensity hypothesis, $\partial y_I/\partial k < 0$. Therefore, the sum of the characteristic roots is negative while the product of the roots is positive. Thus the nontrivial equilibrium (k^*,p^*) is shown to be locally stable. Furthermore, since the nontrivial equilibrium is stable, it is unique.[2]

The full dynamics of the system (2.6) - (2.7) are depicted in the phase diagram of Figure 2. Incomplete specialization prices $\underline{p}(k)$ and $\bar{p}(k)$--the absolute values of the respective slopes of the PPF at the corner $y_I = 0$ and the corner $y_c = 0$---are indicated as increasing functions of k.[3] Above

---

[1] See [8], pp. 337-338. In the notation of [14], $y_I = \frac{k_c - k}{k_c - k_I} f_I(k_I)$ so that $\frac{\partial y_I}{\partial k} \frac{-f_I(k_I)}{k_c - k_I} < 0$ when $k_c > k_I$.

[2] And therefore when a nontrivial equilibrium exists, the trivial equilibria (with $k = 0$) are not stable.

[3] That $\underline{p}(k)$ and $\bar{p}(k)$ are increasing functions of k when the factor intensity hypothesis is satisfied is proved by Oniki and Uzawa [6] using the techniques developed by Johnson [5].

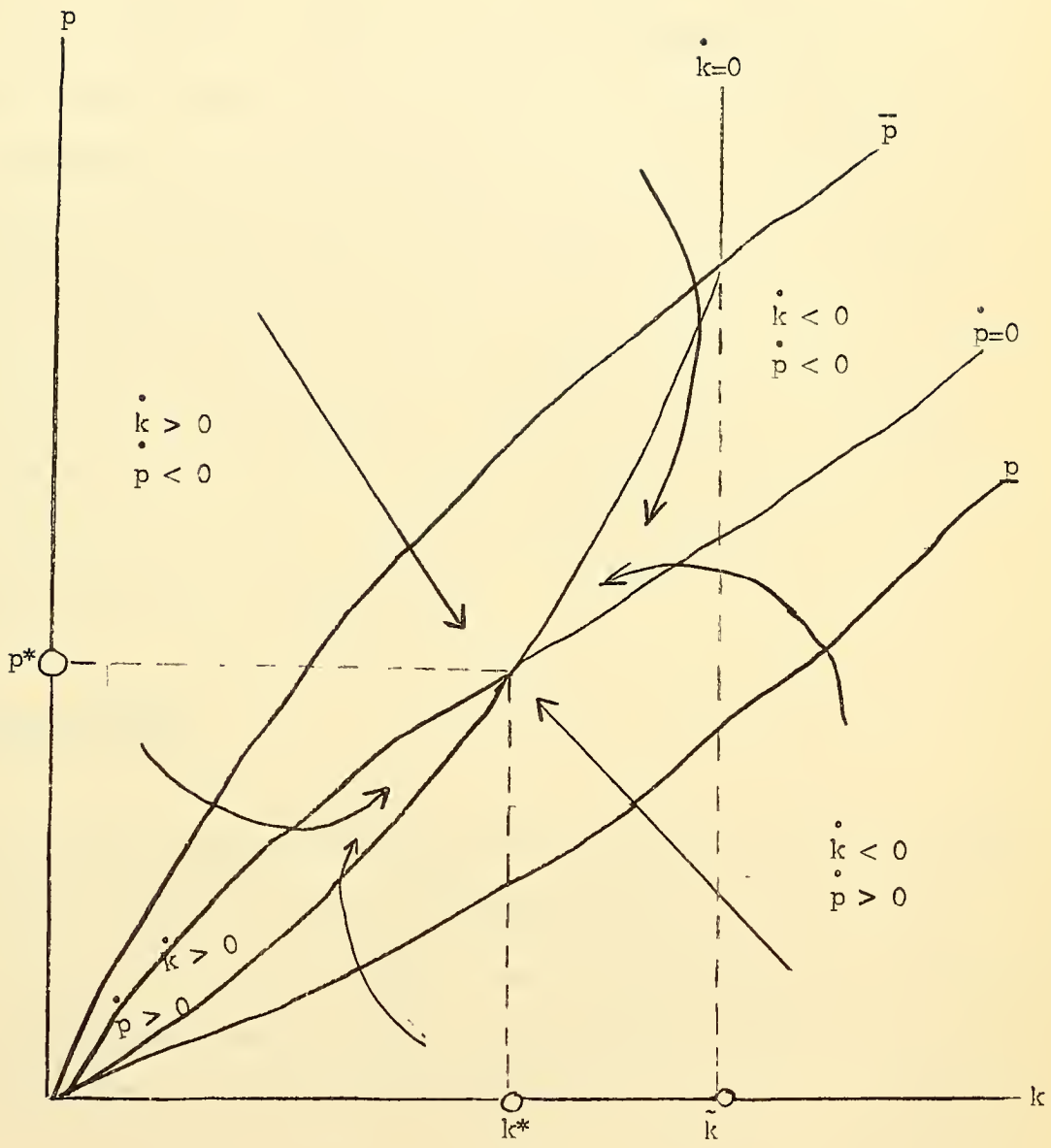
p
$\dot{k}=0$
$\bar{p}$
$\dot{k} < 0$
$\dot{p} < 0$
$\dot{p}=0$
$\dot{k} > 0$
$\dot{p} < 0$
$\underline{p}$
p^*
$\dot{k} < 0$
$\dot{p} > 0$
$\dot{k} > 0$
$\dot{p} > 0$
k
k^*
$\tilde{k}$

Figure 2

$\bar{p}(k)$ the economy specializes to production of the capital good, by (2.6) $\dot{p} < 0$, and by (2.7) $\dot{k} \gtrless 0$ as $k \lessgtr \tilde{k}$, where $\tilde{k}$ is the maximum sustainable capital-labor ratio defined by $f_I(\tilde{k}) = n\tilde{k}$. Below $\underline{p}(k)$ the economy specializes to the production of the consumption good, so that $\dot{k} < 0$ and $\dot{p} > 0$. The $\dot{p} = 0$ curve must, of course, always lie between $\underline{p}(k)$ and $\bar{p}(k)$. To the left of $\tilde{k}$, the $\dot{k} = 0$ curve must lie between $\underline{p}(k)$ and $\bar{p}(k)$ but at $\tilde{k}$, the $\dot{k} = 0$ locus must cut the $\bar{p}(k)$ curve. Consequently at the unique nontrivial equilibrium (k^*,p^*), the positive slope of the $\dot{k} = 0$ curve must be greater than the positive slope of the $\dot{p} = 0$ curve. Thus, as is shown in Figure 2, economies initially endowed with a positive capital-labor ratio tend asymptotically to the (k^*,p^*) equilibrium.

3. The One-Sector Model

We can also study the implications of our reformulation of the neo-classical savings hypothesis in terms of the Solow-Swan, one-sector model (which is a special case of the two-sector model with equal factor intensities). Momentary equilibrium requires that the value of savings be a constant fraction of Individual Purchasing Power $\hat{Y}$.

$$CG + p\dot{K} = s\hat{Y}, \tag{3.1}$$

where p is the market consumption price of capital and expected capital gains CG are equal to $p\dot{K}$ under the assumption that expectations about price

changes are fulfilled. $\hat{Y} = Y + CG$, where $Y = \max(1,p)Q$ is the consumption value of output, and Q is the quantity of output in physical units. Setting $CG = \dot{p}K$ yields

$$s[\max(1,p)Q + pK] = \dot{p}K + p\dot{K}, \tag{3.2}$$

or

$$(1 - s)[\max(1,p)Q + \dot{p}K] = C, \tag{3.3}$$

where C denotes consumption. If $p < 1$, the economy is specialized to the production of the consumption good which implies, in the absence of depreciation, that $\dot{K} = 0$. Hence, for (3.2) to hold when $p < 1$, capital gains must be positive, $\dot{p} > 0$. On the other hand, $p > 1$ implies that the economy is specialized to the production of the investment good, $C = 0$. Hence from (3.3) $p > 1$ implies that $\dot{p} < 0$. Letting $Q/L = f(k)$, we have that

$$\left\{ \begin{array}{l} \dot{k} = -nk \\ \\ \dot{p} = \dfrac{s}{1-s}\dfrac{f(k)}{k} \end{array} \right\} \quad \text{for } p < 1 \tag{3.4}$$

and

$$\left\{ \begin{array}{l} \dot{k} = f(k) - nk \\ \\ \dfrac{\dot{p}}{p} = \dfrac{-f(k)}{k} \end{array} \right\} \quad \text{for } p > 1 \tag{3.5}$$

If $p < 1$, k is falling so that the average product of capital $(f(k)/k)$ is rising[1] and thus from (3.4) we know that p is increasing faster than a constant absolute rate so that in finite time $p = 1$. If $p > 1$, $(f(k)/k)$ is either rising or bounded from below by $(f(\tilde{k})/\tilde{k})$ where $\tilde{k}$ is the maximum sustainable capital-labor ratio defined by $f(\tilde{k}) = n\tilde{k}$. Hence from (3.5) we know that when $p > 1$, p falls faster than a constant relative rate. That is, if initially p is different from unity, p will become unity in finite time.

What happens when p is equal to unity? Since

$$\dot{k} = sf(k) - (1-s)p\dot{k} - nk, \qquad \text{for } p = 1, \tag{3.6}$$

every point of the PPF is a possible momentary equilibrium. If $\dot{p} > 0$ (or < 0) for more than "a moment," p will then differ from unity. But we know that as soon as p differs from unity, it moves toward unity. Assuming, as we have throughout, that p is a continuous function of time, then it follows that if $p = 1$, then $\dot{p} \neq 0$ only for isolated moments. Thus, although momentary equilibrium is not unique when $p = 1$, the system is causal: If ever p is equal to unity, it will remain at unity and $\dot{k} = sf(k) - nk$ except on a set of measure zero.

If $p = 1$ and $\dot{p} = 0$, there are no capital gains and the economy behaves just as Solow described it. That is,

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[1] Assuming that $f'(k) > 0$ while $f''(k) < 0$.

$$\frac{f(k)}{k} \to \frac{n}{s} \quad \text{as } t \to \infty.$$

Thus, in the one-sector model the unique long-run balanced growth equilibrium is globally stable.

Government Debt. In the simple two-commodity closed economy, the possibility of specialization is little more than a curiosum. Therefore, Solow's omission of prices (and thus capital gains) in the one-sector model was not unjustified. On the other hand, if there exists a second store of value, for example, government debt, the possibility of zero investment is no longer a mere curiosum.

Again, we take consumption as the numeraire so that

$$\dot{W} = p\dot{K} + p_B\dot{B} + CG, \tag{3.7}$$

where B (for bonds) is the nominal stock of government debt and p_B is the consumption price of a bond. If CG denotes appreciation in the consumption value of assets, then

$$CG = \dot{p}K + \dot{p}_B B. \tag{3.8}$$

Assume that the nominal rate of interest on government bonds iz zero.[1]

[1] Indeed, the reader may think of this noninterest bearing debt as "money" in an economy in which there are no transactions nor liquidity preference demands for money.

Then, for the asset market to be in momentary equilibrium, it is required that both assets yield the same rate of return, i.e.,

$$\frac{\dot{p}}{p} + \frac{\max(1,p)f'}{p} = \frac{\dot{p}_B}{p_B} . \tag{3.9}$$

Equation (3.9) states that both assets must have the same rental plus price appreciation per consumption unit.[1] If both goods are produced, then the market price of capital must equal the market price of output which in turn must equal the market price of consumption, i.e., $p = 1$. Let $\theta = \dot{B}/B$ be the increase in the nominal supply of bonds and $b = p_B B/L$ be the consumption value of the per capita stock of bonds. Then from (3.7) and (3.8), if we assume[2] that the consumption value of the community's savings (change in the value of wealth) is a constant fraction s of the consumption value of Individual Purchasing Power[3] (output plus the change in the value of wealth),

---

[1] Our model is easily extended to the case where bonds are more liquid than capital. Then the demand for bonds would depend on the rates of return on bonds as well as on capital, but then, in general, equation (3.9) would not hold.

[2] We assume throughout that individuals instantaneously adjust their expectations about price changes. That is, expected right-hand time derivatives of price are set equal to actual left-hand derivatives. If $p(t)$ is continuously differentiable, then instantaneous adjustment is equivalent to short-run perfect foresight.

[3] For convenience, we assume that when the government creates new bonds, these bonds are transferred to the public. Thus, $Y = \dot{p}K + \dot{p}_B B + p_B \dot{B} + \max(1,p)Q$. If both goods are produced, then $p = 1$ and $\dot{p} = 0$.

$$\dot{k}/k = sf(k)/k - (1-s)[\theta + (\dot{p}_B/p_B)]b/k - n.$$

Since we are only considering the case where both goods are produced, $p \equiv 1$, from (3.9) we have that

$$f' = \dot{p}_B/p_B$$

and therefore

(3.10) $$\dot{k}/k = sf(k)/k - (1-s)[\theta + f'(k)](b/k) - n.$$

Logarithmic time differentiation yields

(3.11) $$\dot{b} = [f'(k) + \theta - n]b,$$

from (3.9) and the definition of b.

We analyze the dynamic behavior of the system (3.10) - (3.11) assuming that the government pursues a policy of a constant rate of expansion of the nominal supply of bonds, i.e., the case when θ is a given constant. From (3.11), $\dot{b} = 0$ if and only if $f'(k) = n - \theta$. If the production function is neoclassical and satisfies the Inada conditions, then for $n-\theta > 1$, there exists a unique value k^* of the capital-labor ratio that yields a stationary to (3.11) for $b > 0$, i.e.,

$$f'(k^*) = n - \theta > 0.$$

This is indicated in the phase diagram of Figure 3. For $k > k^*$, b is decreasing, for $k < k^*$, b is increasing. Setting the left-hand side of (3.10) equal to zero and substituting k^* for k shows that the stationary solution (b^*,k^*) to differential equations (3.10) and (3.11) is unique and that b^* is given by

$$b^* = \frac{sf(k^*) - nk^*}{(1-s)n} .$$

If $b = 0$, $\dot{k} = 0$ if $sf(k) = nk$. Therefore, there exists a second nontrivial balanced growth equilibrium[1] which is denoted in Figure 3 by the point $(0,k^{**})$, where $sf(k^{**}) = nk^{**}$.

In what follows, it is assumed that the production function $f(\cdot)$ and the parameters n and θ are such that b^* is positive.[2] Setting $\dot{k} = 0$, implicit differentiation of (3.10) yields

$$\left(\frac{dk}{db}\right)_{k=0}^{\cdot} = \frac{(1-s)[\theta + f'(k)]}{sf'(k) - n - (1-s)bf''(k)} ,$$

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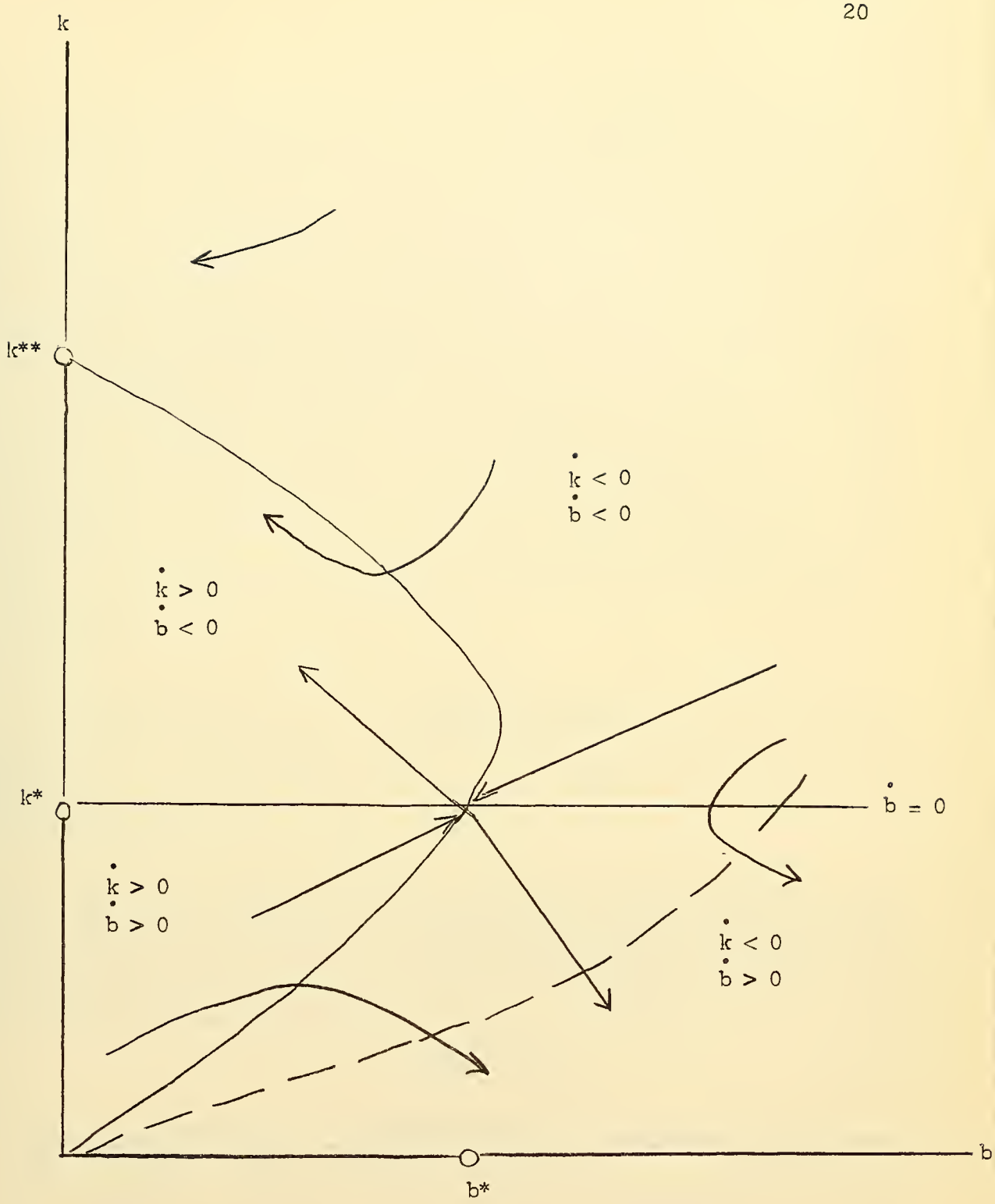
[1] The Solow zero-bonds equilibrium.

[2] It is interesting to notice that if the government sets θ too high (greater than n), then there is no steady state solution in which the public holds bonds.


k
k**
k*
$\dot{k} < 0$
$\dot{b} < 0$
$\dot{k} > 0$
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$\dot{b} = 0$
$\dot{k} > 0$
$\dot{b} > 0$
$\dot{k} < 0$
$\dot{b} > 0$
b
b*

Figure 3

which is unsigned, although under our assumptions the $\dot{k} = 0$ locus intersects the $\dot{b} = 0$ locus exactly once.[1] From Figure 3, we see that to the right of the $\dot{k} = 0$ curve, k is decreasing; to the left of the $\dot{k} = 0$ curve, k is increasing. Thus, the equilibrium (b*,k*) is a saddlepoint. That is, given initial endowments K(0), L(0), B(0) there exists only one initial assignment of the price of bonds $p_B(0)$ that will lead the economy to the nontrivial balanced growth state with bonds, (b*,k*). As in the models of Cagan [1], Sidrauski [11], Hahn [2], and Shell and Stiglitz [10], there is nothing in the model so far presented to ensure that this unique initial price to be "chosen" by the economy.[2] The Solow zero bond equilibrium (0,k**) is, on the other hand, locally stable.

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[1] For $\dot{k}$ to equal zero, $b = \frac{sf(k) - nk}{(1-s)[\theta + f'(k)]}$; $b = 0$ if $sf = nk$, and $\lim_{k \to 0} \left(\frac{b}{k}\right)_{k=0} = 0$. (Observe that along $\dot{k} = 0$ b is single valued in k, but k is not single valued in b.)

[2] Paths not converging to the (b*,k) equilibrium either (a) converge to the Solow (0,k**) balanced growth equilibrium or (b) in finite time have such large capital gains that real investment goes to zero. In Figure 3, we have indicated by a dashed curve the locus of points along which all of output is consumed while p = 1. To the right of the dashed curve p must be less than unity. Along the dashed curve

$$b = \frac{sf(k)}{(1-s)[\theta + f'(k)]}$$

and

$$\frac{db}{dk} = \frac{sf'(k)}{(1-s)[\theta + f'(k)]} - \frac{sf(k)f''(k)}{(1-s)[\theta + f'(k)]^2} > 0 \text{ for } 0 < k < k^* \text{ or } \theta \geq 0.$$

Consider a trajectory crossing the dashed line from the left with p = 1. The asset market clearing equation is $\dot{p}_B/p_B = f'/p + \dot{p}/p$ and therefore the savings-investment equation yields

Two important features of this economy distinguish it from our extensions of the Solow one-sector model and the Uzawa two-sector model. First, the long-run equilibrium (b^*,k^*) is a saddlepoint. This is a property that the government-debt model shares with any competitive model with more than one asset and an asset market clearing equation consistent with the hypothesis that individuals instantaneously adjust their expectations about price changes.[1] Second, even when there is no increase in the nominal supply of bonds, i.e., when $\theta = 0$, in the government-debt economy the inclusion of asset appreciation implies that the long-run equilibrium capital-labor ratio will be less than in the corresponding no-bond economy.[2]

$$\dot{p} = \frac{[p\, sf/(1-s)b - \theta] - f'}{pk/b + 1}$$

With $p < 1$, $\dot{k}/k = -n < 0$ and in finite time $k < k^*$ so that in finite time $f'/p + \theta - n > f' + \theta - n > 0$. Since $\dot{b}/b = \theta - n + f'/p + \dot{p}/p$, $\frac{d(pk/b)}{dt} < 0$. So $p > 1$ falls faster than at a constant absolute rate, and thus the price of capital goes to zero in finite time.

But if capital is freely disposable when $p = 0$, $\dot{p} \geq 0$. Since $f' > 0$, the rate of return on capital is then infinite. Remember that asset market clearance requires that $\dot{p}/p + f'/p = \dot{p}_B/p_B$. So with $p = 0$ and $p_B > 0$, p and p_B must be discontinuous and expectations about price changes must be frustrated. For a detailed analysis of this type of problem and its implications for competitive growth, see Shell and Stiglitz [10].

[1] See Hahn [2] and Shell and Stiglitz [10].

[2] So "government debt matters" in the sense that its inclusion affects the long-run equilibrium capital-labor ratio, the dynamic stability of the system and, of course, the adjustment path. See also Tobin [13] and Sidrauski [11].

4. Further Implications of the Individual Purchasing Power Savings Hypothesis

Units in Which Purchasing Power is Reckoned. Unlike most of the neoclassical growth models, many of the important qualitative and quantitative properties of the models treated in this paper depend crucially upon the units in which our concept of "income" is defined. For example, in the one- and two-sector models, if the change in the capital goods value of wealth is assumed to be a constant fraction of the capital goods value of Purchasing Power, then development of the economy will be very different from that described in Sections 2 and 3. Indeed, if Purchasing Power is reckoned in capital goods units, then since capital gains are zero, the stories will be just as Solow and Uzawa told them.

On the other hand, in our economy with government debt, since there are two assets, there is no choice of Purchasing Power unit that will eliminate the phenomenon of asset appreciation. Consider cases in which the price of the consumption good is equal to the price of the capital good. Since B is noninterest bearing government debt, we can think of it as "money." If Purchasing Power is reckoned in consumption (or capital goods) units, then

$$C = (1-s)\left[Q + \dot{p}_B B + p_B \dot{B}\right]. \tag{4.1}$$

If, however, Purchasing Power is reckoned in money units, then

$$\pi C = (1-s)\,\pi Q + \dot{B} + \pi K\ , \tag{4.2}$$

where π is the money price of output (equal to the money prices of consumption and capital). Notice that these two specifications of the consumption function are fundamentally different. Under specification (4.1), the larger the rate of inflation (i.e., the smaller $\dot{p}_B$), the smaller is consumption C. While under specification (4.2), the larger the rate of inflation $\dot{\pi}$, the larger is consumption C.

In what units do individuals reckon Purchasing Power? Money is the most convenient unit of accounting. In a world with many commodities and many assets, individuals may find it convenient to follow the simple decision rule: Save a given percentage of the money value of Purchasing Power. But, of course, following such a rule will imply money illusion.[1]

Depreciation. In their survey article, Hahn and Matthews [3] have called attention to the problems raised by depreciation: Are gross savings a constant fraction of gross income or are net savings a constant fraction of net income? It should be clear that the behavior of the economy in the short run and in the long run depends upon which of these assumptions is chosen. For example, in the one-sector model when the price of consumption is equal to the price of capital, the "net-net" hypothesis yields

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[1] Assume, for instance, that there is no increase in the nominal money supply, $\dot{B} = 0$. Increase once-and-for-all the price of commodities π, then under specification (4.2) consumption will go up or down depending upon whether $\dot{\pi}$ is negative or positive. But under specification (4.1), if the price of commodities is increased (p_B is decreased) once-and-for-all, then C remains unchanged, because in consumption units Purchasing Power is unchanged.

$$\dot{k} = s[f(k) - \mu k] - nk.$$

If population is not growing, $n = 0$, the "net-net" economy asymptotically tends to zero consumption. And it is this "net-net" assumption which agrees with our hypothesis that the net change in wealth is a constant fraction of Purchasing Power.[1]

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[1] If $n = 0$ and $\mu > 0$, then $\lim_{t\to\infty} k(t)$ is bounded above the golden rule capital-labor ratio. In the no population growth case, therefore, the stylized economy is dynamically inefficient. See Phelps [7]. Since the case $n = 0$ is not empirically interesting, we should neither jump to the conclusion that the Individual Purchasing Power savings hypothesis is an incorrect specification nor to the conclusion that real world capitalist economies do a poor job in the intertemporal allocation of resources.

Indeed, if pushed to empirically uninteresting extremes, our model can be made to reveal other unrealistic curiosa. For example, if initially the price of investment is greater than the incomplete specialization price, our economy would go for a time with zero consumption. The point is that this is a model in which savings-consumption decisions are made according to fixed rules. Whether or not our formulation is a superior description is a question of fact to be subjected to further empirical test based upon actual historical evidence.

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